Importance of precise geoid model in direct georeferencing and aerial photogrammetry:
A case study in Sweden

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6th April 2022

Special thanks to: Anders Ekholm, Håkan Ågren and Kulla Rolf (Lantmäteriet)

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Background and Aim

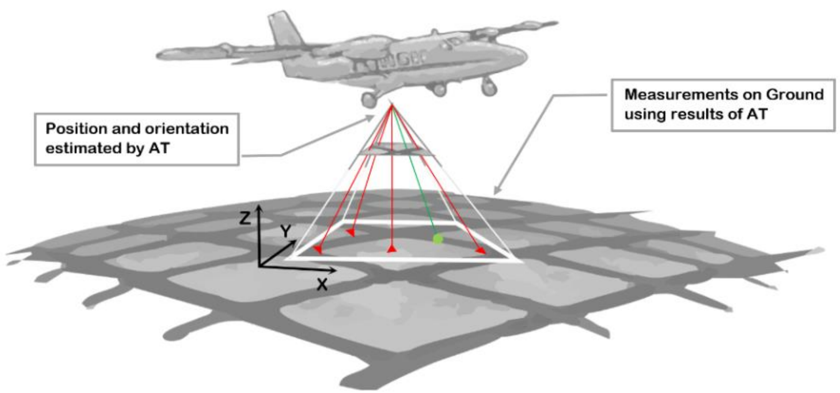
Background: Collaboration between Lantmäteriet and HiG:

1. Automated decision making
2. Information supply in Geodata area (Change detection, image analysis, 3D modelling, BIM and crow-sourcing)
3. Information presentation and visualization

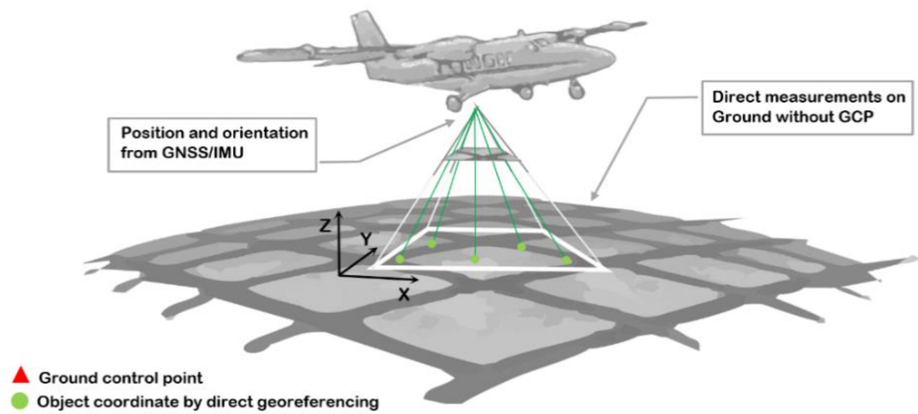
PhD thesis Title: Quality assessment in 3D mapping using aerial photogrammetry data

One of the thesis objectives: Analyzing the influence of the deflection of verticals (DOV) in direct georeferencing (DG) of collected aerial images

Classical aerial triangulation using GCPs

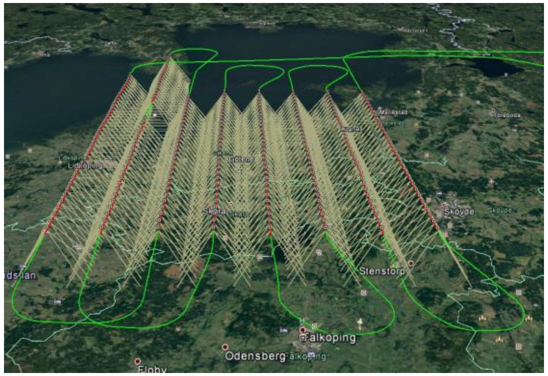


Direct georeferencing (DG)



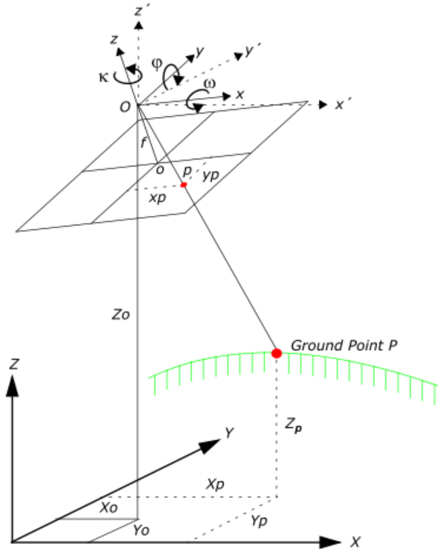
Backgroud

1



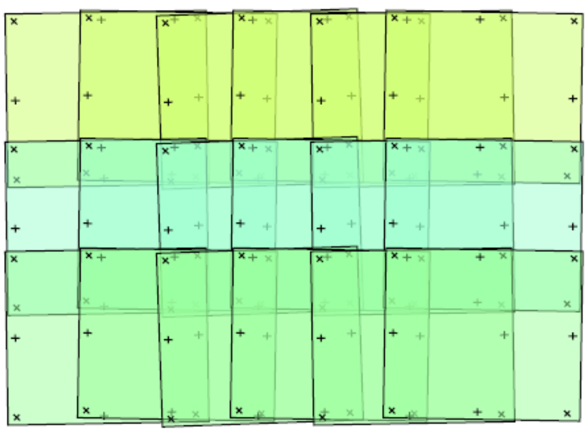
Data collection

2



Exterior Orientation Paprameters

3



Aerial Triangulation

4

Orthophoto

DEM

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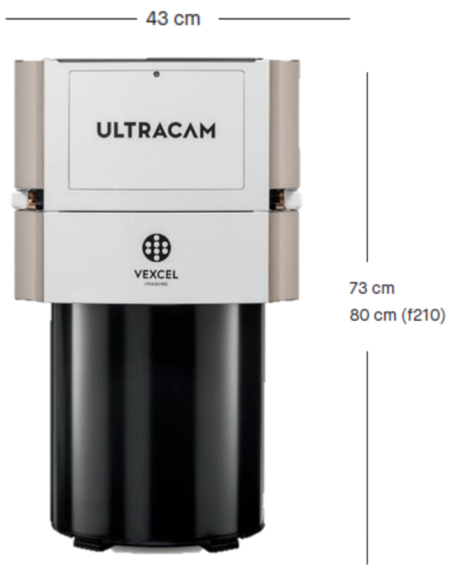
Background

- Applanix POS AV 510 – GNSS/INS Equipment

The data-acquisition equipment specifications

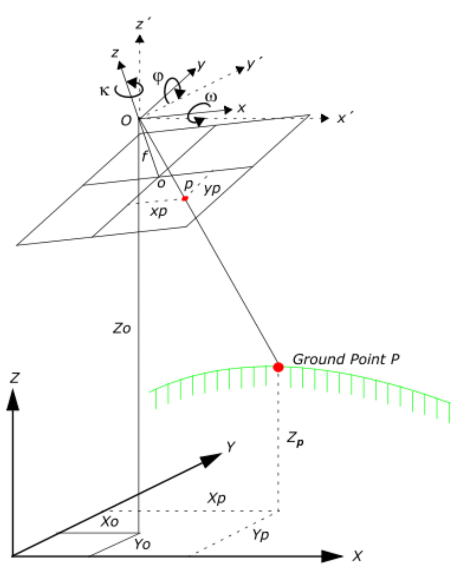
Camera		UltraCam Eagle, c=80mm
GNSS receiver		Applanix GPS17 (L1, L2)
Data rate		1 sec
IMU		Applanix IMU31
Data rate		200 Hz
Noise		0.02 deg/sqrt(hr)
IMU drift		0.1 deg/hr
GNSS/IMU system		Pos / AV 510-DG
claimed accuracy	*Position*	< 0.1m
	Velocity	< 0.005 m/s
	Roll, Pitch	< 0.005 deg.
	Yaw	< 0.008 deg.

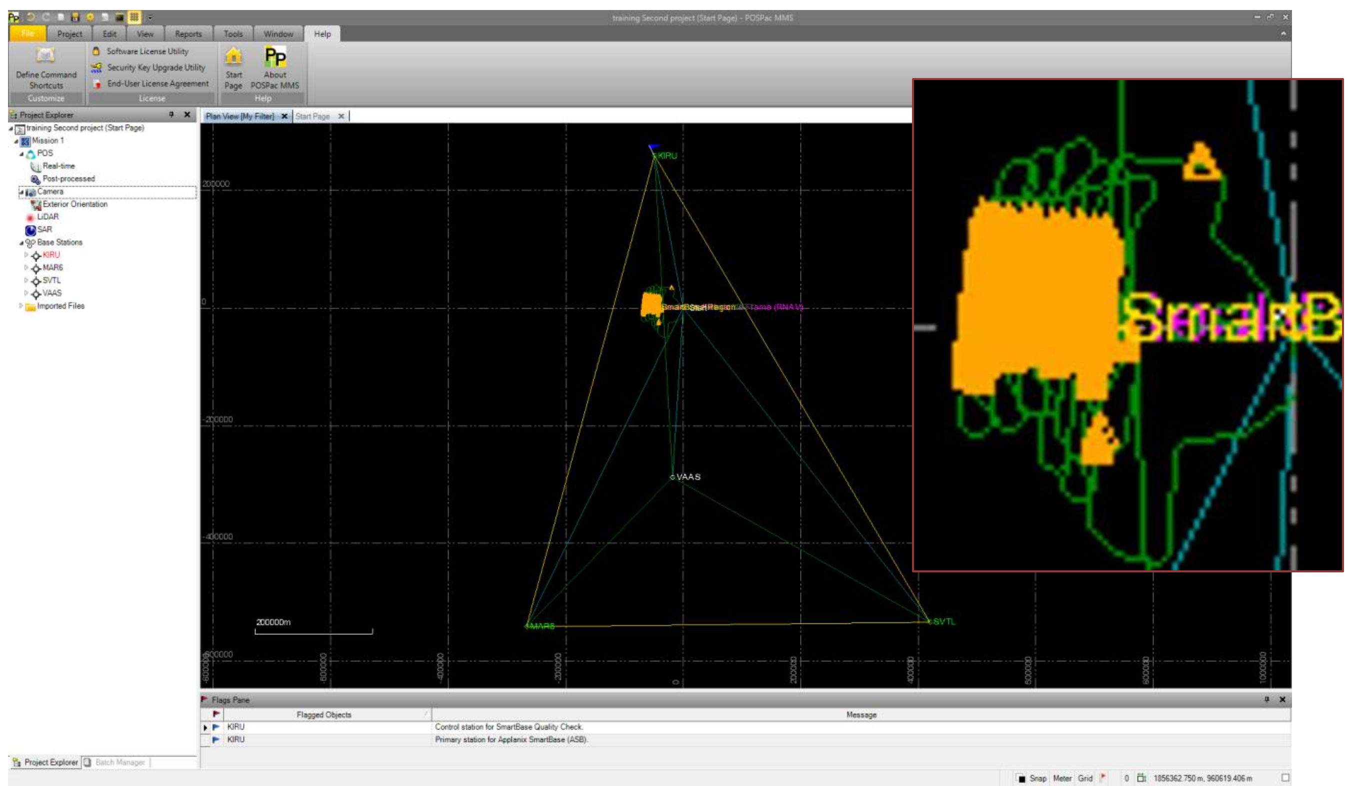
- Ultracam Eagle digital camera with
  - 80 mm lens
  - 67° (46.1°) camera field of view (FOV) in across track (along track).



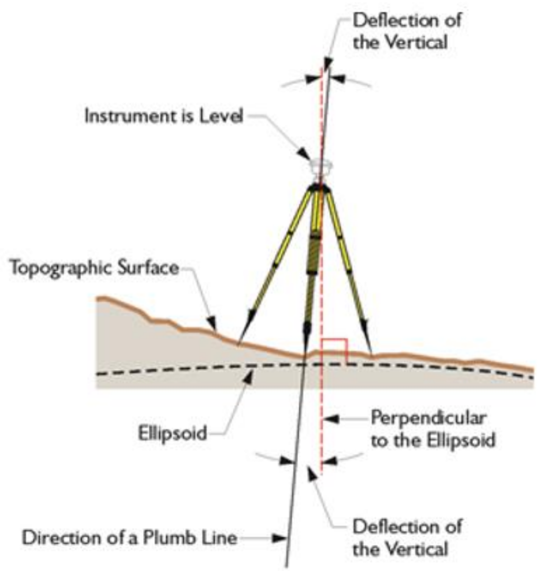
Background

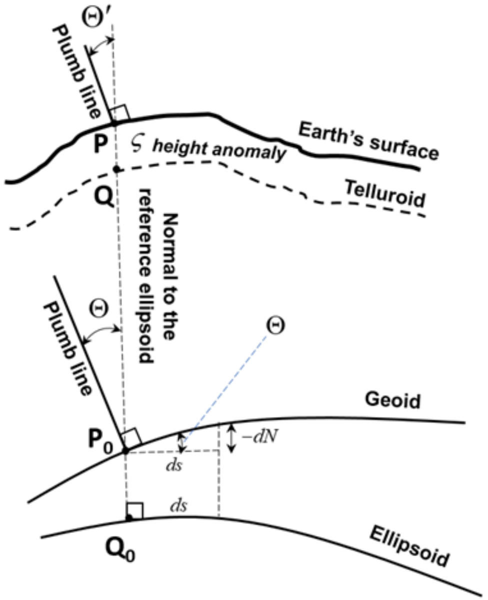
- **Applanix POSPac** is a software to process the collected GNSS and IMU data in Direct Georeferencing.
- Exterior orientation parameters (EOPs) are obtained by integrating GNSS and IMU data using Kalman filtering in **POSPac** software (E, N, H, fi, Omega , kappa).





Problem definition: Deflection of verticals

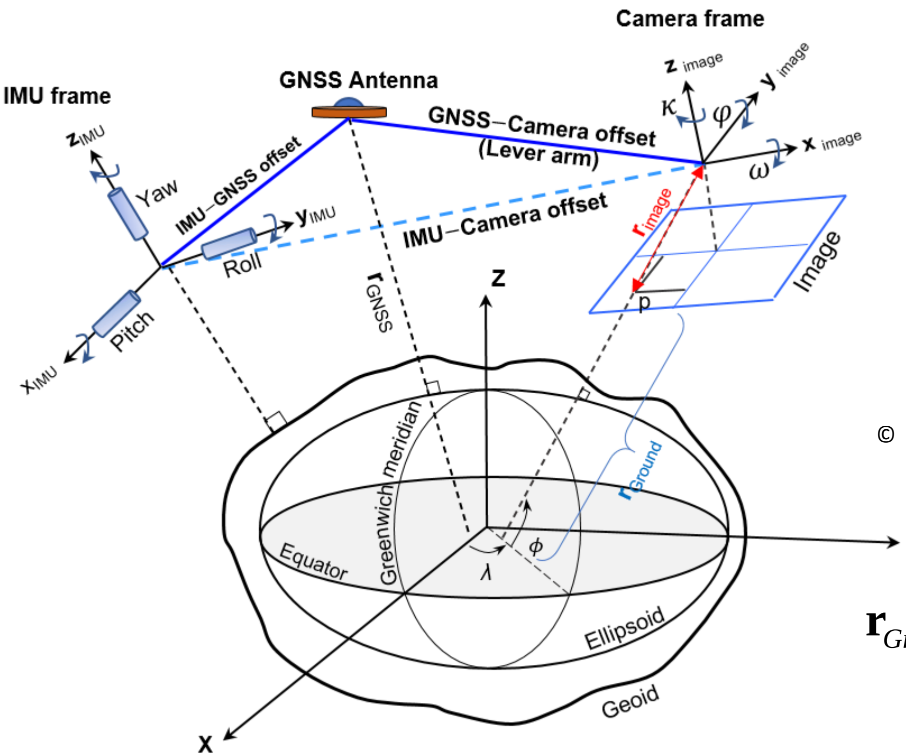




Deflection of the plumb line and normal to the ellipsoid.

The deviation between the gravity vector and the ellipsoidal normal at a point is called deflection of the vertical.

Problem definition



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$$\mathbf{r}_{Ground} = \mathbf{r}_{GNSS} + \mathbf{R}_{DOV}\mathbf{R}_{INS}\left(\mathbf{r}_{lever\ arm} + s\mathbf{R}_{Boresight}\mathbf{r}_{image}\right)$$

Different angular systems in aerial photogrammetry
(Inertial navigation solution provides roll, pitch and heading whereas the photogrammetric system uses(ω,ϕ,κ) angles).

$\mathbf{R}_{Boresight}$ is the rotation matrix from the camera frame to the aircraft body frame by the boresight angles.

S: is scale factor

$\mathbf{r}_{lever\,arm}$ is the vector of distance between the phase centre of the GNSS antenna and the camera principal point

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Background

INT. J. REMOTE SENSING, 1996, VOL. 17, NO. 11, 2185–2200

Georeferencing of airborne laser altimeter measurements

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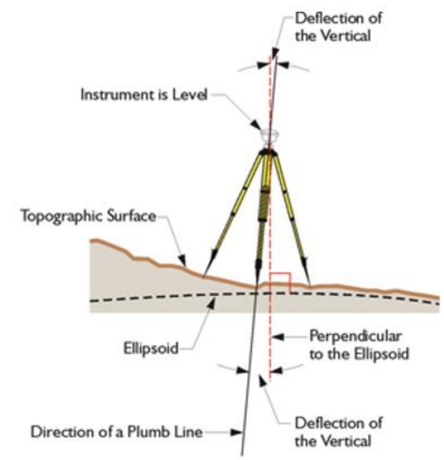
D. RABINE
Science Systems Applications Inc., NASA, Goddard Space Flight Center, Bldg. 22, Greenbelt, MD 20771, U.S.A.

- **DOV effect ignored**
- **Gyroscope accuracy was not accurate in the 1990s (~180“ >> DOV).**



Deflection components and their effects on horizontal and vertical components

- The north-south and east-west directions, the deflection of vertical (DOV) components become (Heiskanen and Moritz, 1967) :

North-south $$\xi = -\frac{1}{R}\frac{\partial \varsigma}{\partial \theta} - \frac{\Delta g}{\gamma}\frac{1}{R}\frac{\partial H}{\partial \theta}$$

East-west $$\eta = -\frac{1}{R\sin\theta}\frac{\partial \varsigma}{\partial \lambda} - \frac{\Delta g}{\gamma}\frac{1}{R\sin\theta}\frac{\partial H}{\partial \lambda}$$

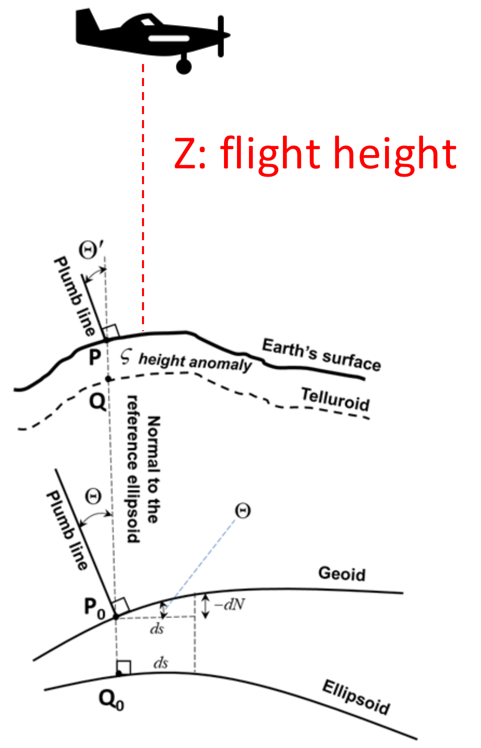
: at the Earth's surface

- Upward continuation to flight altitude **z** (km)

$$\mathbf{r}_{Ground} = \mathbf{r}_{GNSS} + \mathbf{R}_{DOV}\mathbf{R}_{INS}\left(\mathbf{r}_{lever\ arm} + s\mathbf{R}_{Boresight}\mathbf{r}_{image}\right)$$

$$R_{DOV} = R(\eta)R(\xi)$$

$$R_{DOV} = \begin{bmatrix} \cos\eta & \sin\xi\sin\eta & \sin\eta\cos\xi \\ 0 & \cos\xi & -\sin\xi \\ -\sin\xi & \cos\eta\sin\xi & \cos\xi\cos\eta \end{bmatrix}$$



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Deflection components and their effects on horizontal and vertical components

- Impact of DOV on horizontal and vertical components

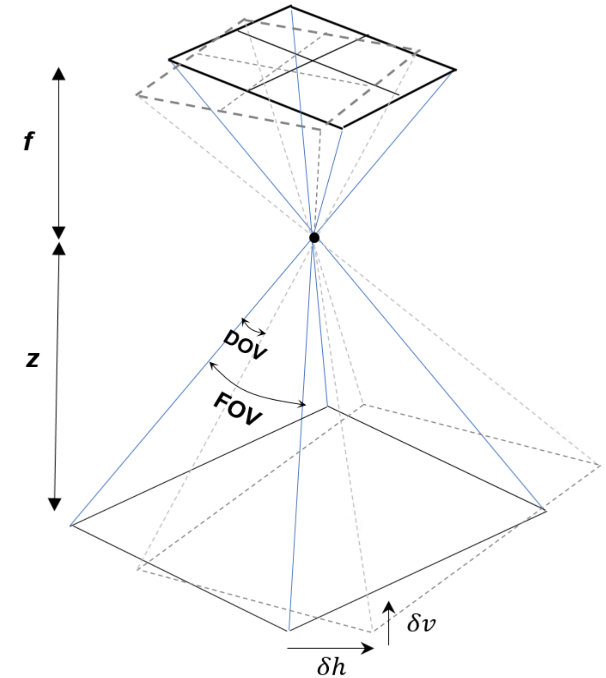
$$\delta h = z \sin(DOV)$$

$$\delta v = z \tan\left(\frac{FOV}{2}\right) \sin(DOV)$$

$$DOV = \xi \cos\alpha + \eta \sin\alpha$$

where:

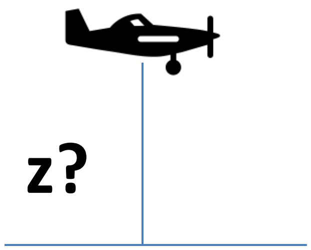
z is the flight altitude, **FOV** is the camera field of view, **α** is azimuth (direction of flight), and **DOV** is the component of deflection of the vertical in the vertical plane orthogonal to the flight direction.



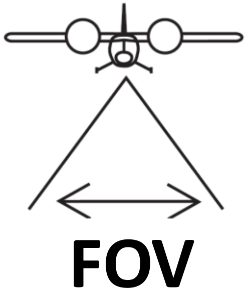
Investigating impact of other parameters

$$\delta h = z\sin(\xi\cos\alpha + \eta\sin\alpha)$$

$$\delta v = z\tan\left(\frac{FOV}{2}\right)\sin(\xi\cos\alpha + \eta\sin\alpha)$$

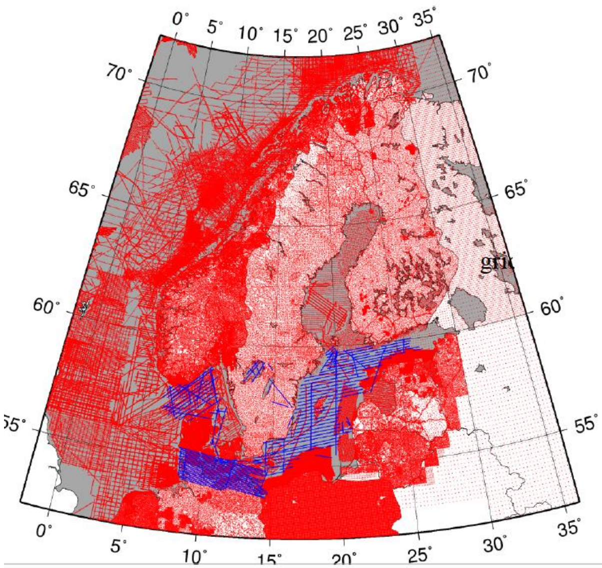


- **Study on the effects of**
  - Flight altitude
  - Camera Field of view (FOV)
  - Flight direction (Azimuth)
  - EGM2008 vs regional data



on the final horizontal and vertical coordinates.

EGM2008 vs regional data

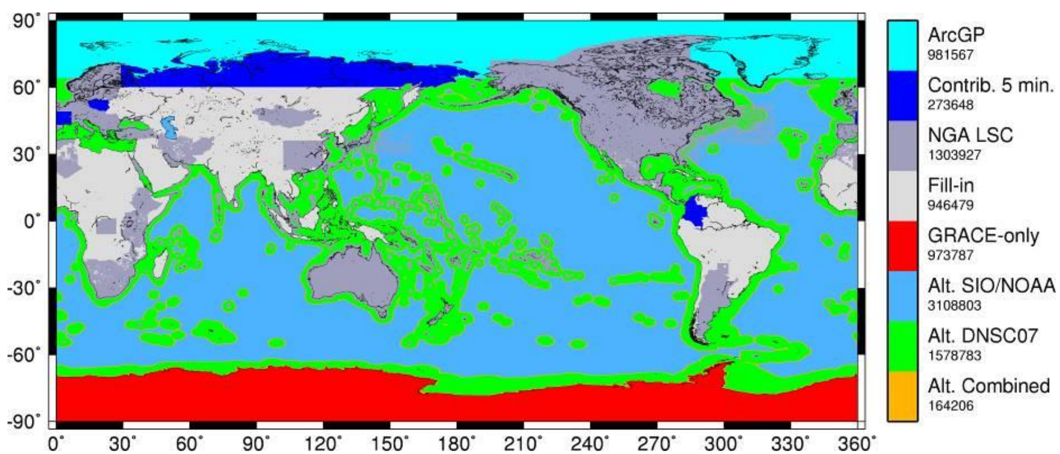


Nordic geodetic commission (NKG) gravity database

Resolution:

0.01° x 0.02° or

0.6' x 1.2' arc min



EGM2008 database

Resolution:

5' x 5' arc min

Results

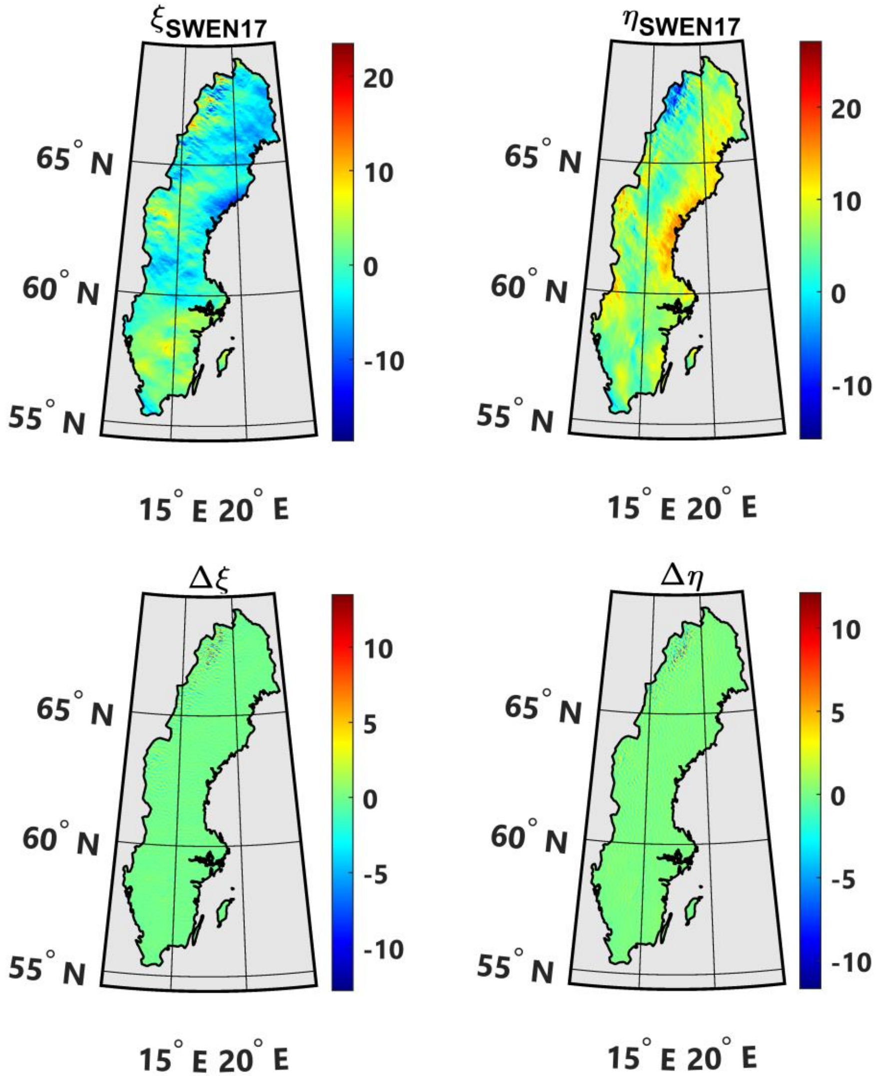
Results: DOV components using EGM2008 and SWEN17 at the Earth's surface

The DOV is also called SWEN17 to follow the same name as the latest geoid model of Sweden i.e. SWEN17_RH2000.

Difference between SWEN17 and EGM2008 models

$$\Delta\xi = \xi_{\mathrm{SWEN17}} - \xi_{\mathrm{EGM2008}}$$

$$\Delta\eta = \eta_{\mathrm{SWEN17}} - \eta_{\mathrm{EGM2008}}$$



Unit: arc second

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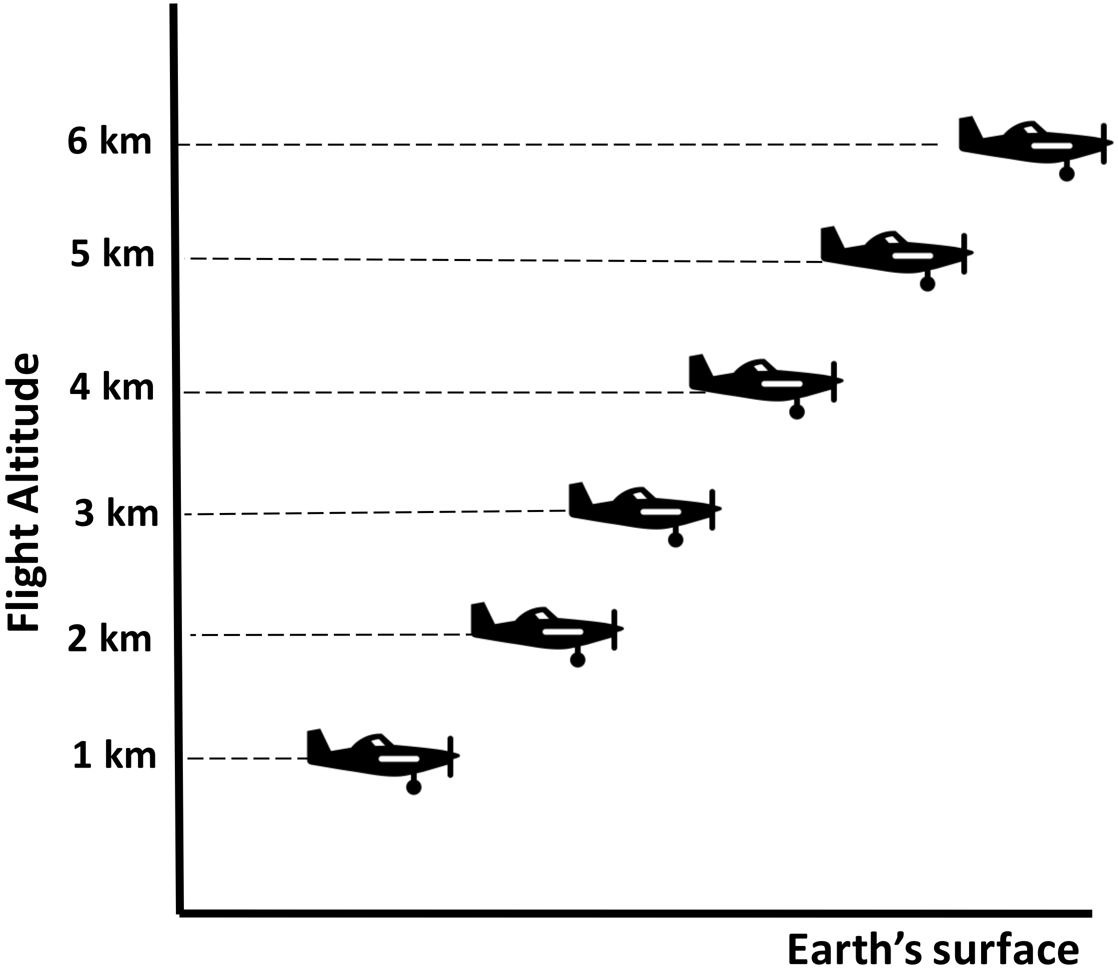
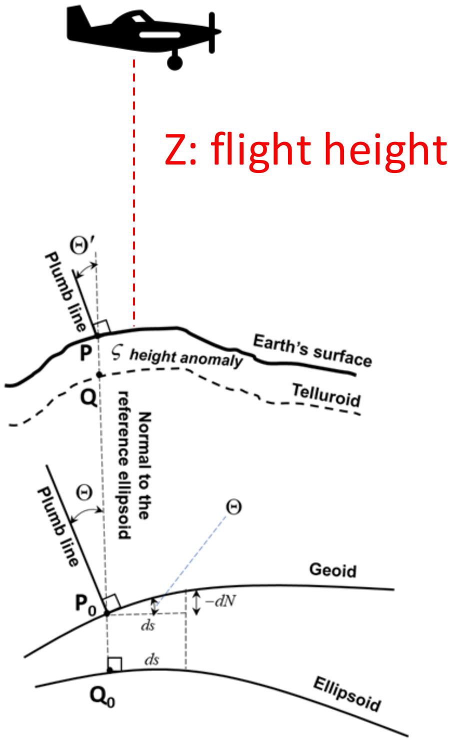
Results: A comparision between DOV components using EGM2008 and SWEN17 at the Earth's surface

Statistics of deflection of the vertical (DOV) using SWEN17 and EGM2008 models and their differences (denoted by Δ) in Sweden.
Unit: arc second.

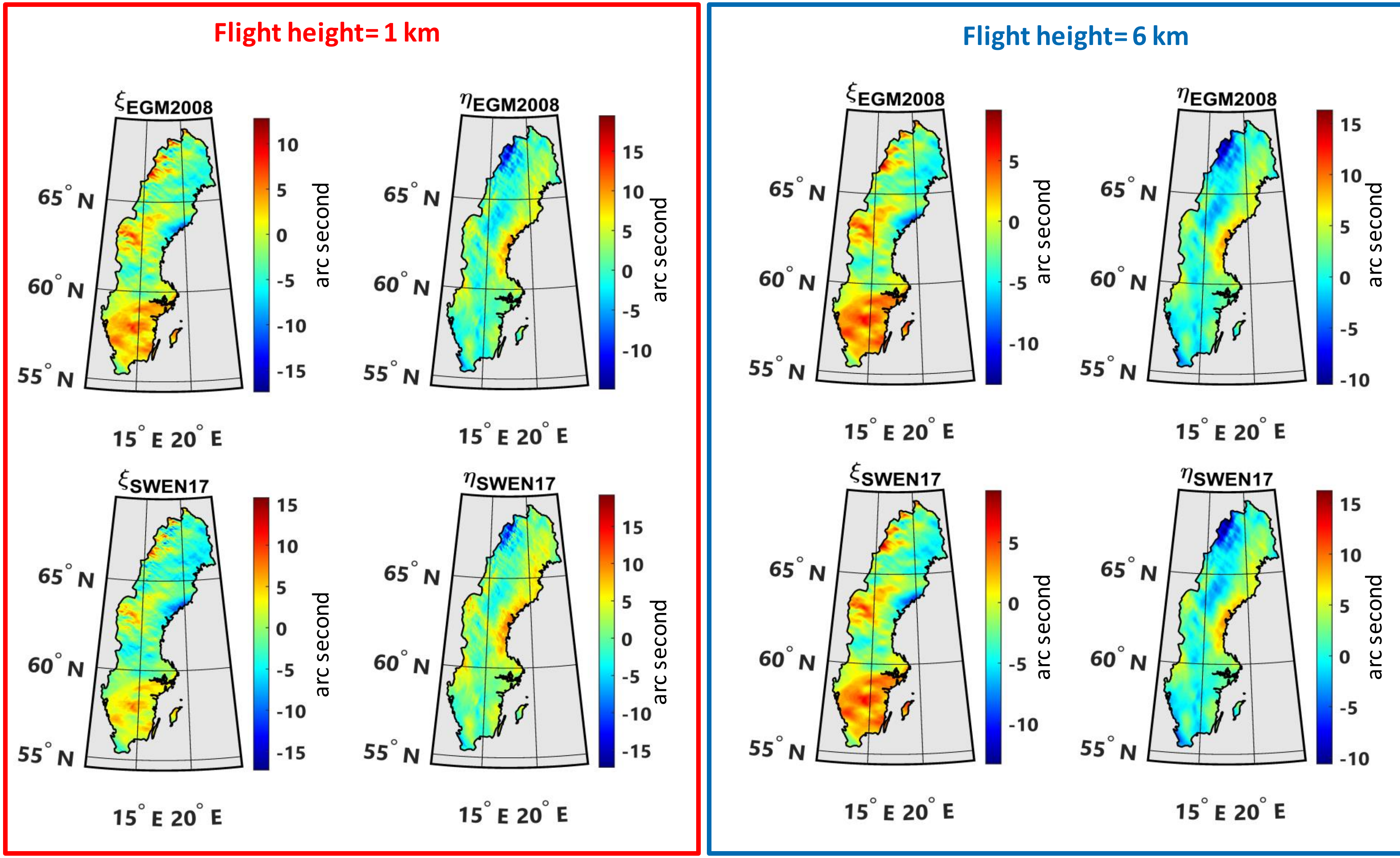
	Max	**Mean**	**Min**	**STD**
$\xi_{EGM2008}$	14.7	0.4	-18.9	3.9
$\eta_{EGM2008}$	25.8	6.2	-10.9	3.6
ξ_{SWEN17}	23.5	0.4	-18.6	3.8
η_{SWEN17}	27.1	6.2	-15.9	4.1
$\Delta\xi$	12.9	0.0	-13.5	1.1
$\Delta\eta$	12.1	0.0	-11.7	1.1

Simulation considering different flight heights





Results: DOV comparison using SWEN17 and EGM08 at different flight heights



Deflection components and their effects on horizontal and vertical components

$$DOV = \xi \cos\alpha + \eta \sin\alpha$$

$$\delta h = z \sin(DOV)$$

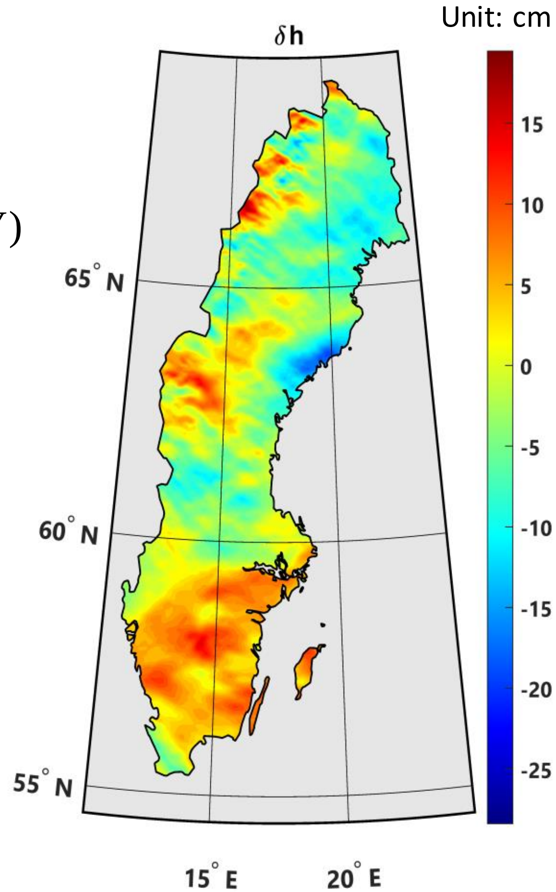
$$\delta v = z \tan\left(\frac{FOV}{2}\right) \sin(DOV)$$

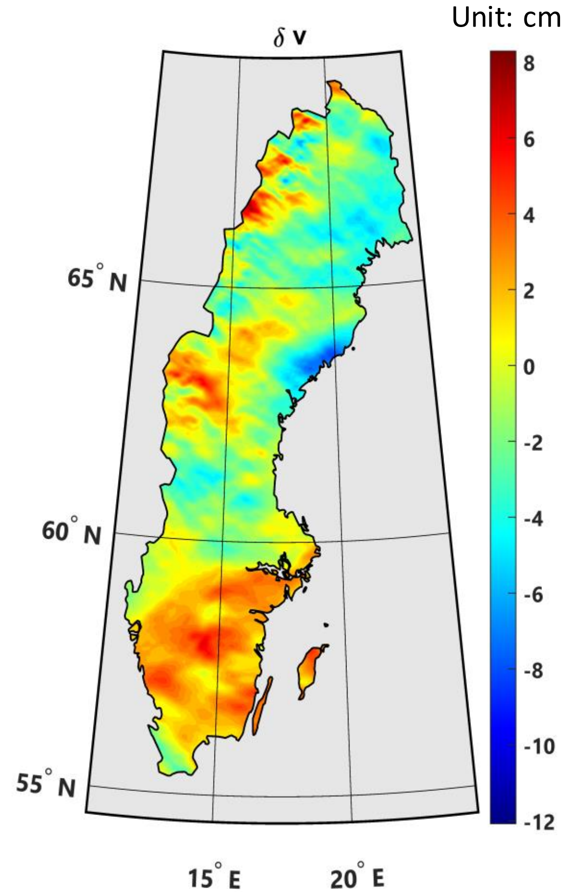
By assuming Azimuth

α= 0˚

FOV= 46.1˚

4km flight altitude





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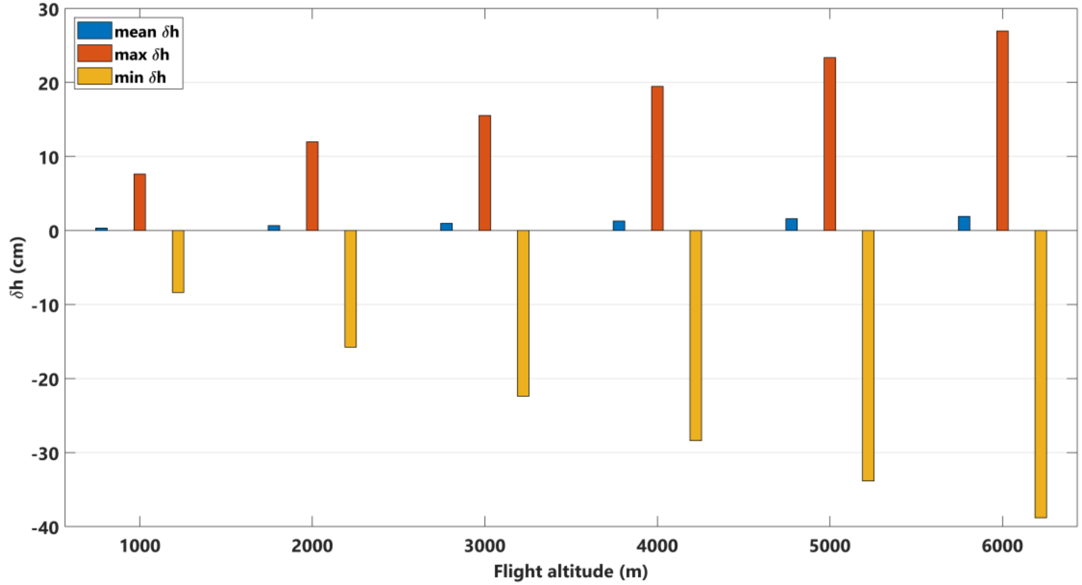
Statistics of the effect of deflection of the vertical (DOV) on horizontal and vertical components using SWEN17

$$\delta h = z\sin(DOV)$$

$$\delta v = z\tan\left(\frac{FOV}{2}\right)\sin(DOV)$$

$$DOV = \xi\cos\alpha + \eta\sin\alpha$$

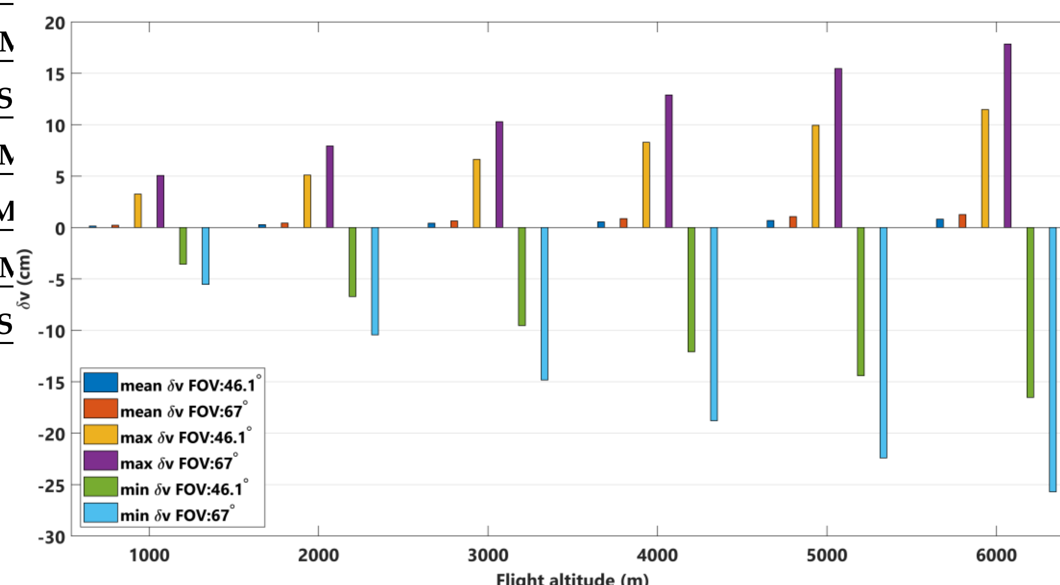


z = 5	z = 6
3.36	26.97
1.60	1.91
33.84	-38.82
7.41	8.68
9.94	11.47
0.68	0.81

Along track

Across track

δv (α=0°) cm	FOV= 46.1°	M
		S
		M
	FOV=67°	M
		M
		S



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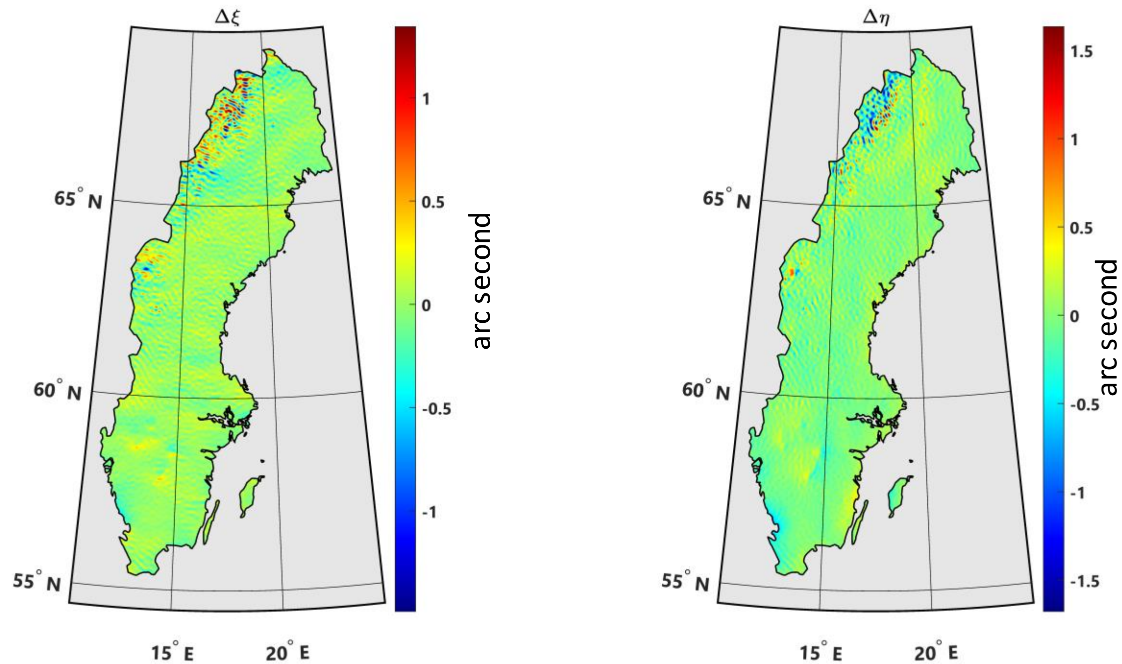
Difference between DOVs

4 km flight altitude

	Max	Mean	Min	STD
$\xi_{EGM2008}$	9.88	0.66	-14.76	3.13
$\eta_{EGM2008}$	17.42	0.69	-11.94	3.51
ξ_{SWEN17}	10.04	0.67	-14.64	3.14
η_{SWEN17}	17.30	0.69	-12.27	3.53
$\Delta\xi$	1.34	0.00	-1.49	0.19
$\Delta\eta$	1.63	0.00	-1.68	0.22

$$\Delta\xi = \xi_{SWEN17} - \xi_{EGM2008}$$

$$\Delta\eta = \eta_{SWEN17} - \eta_{EGM2008}$$



< ±2 arc seconds

The effect DOV anomaly computed using SWEN17and the EGM2008

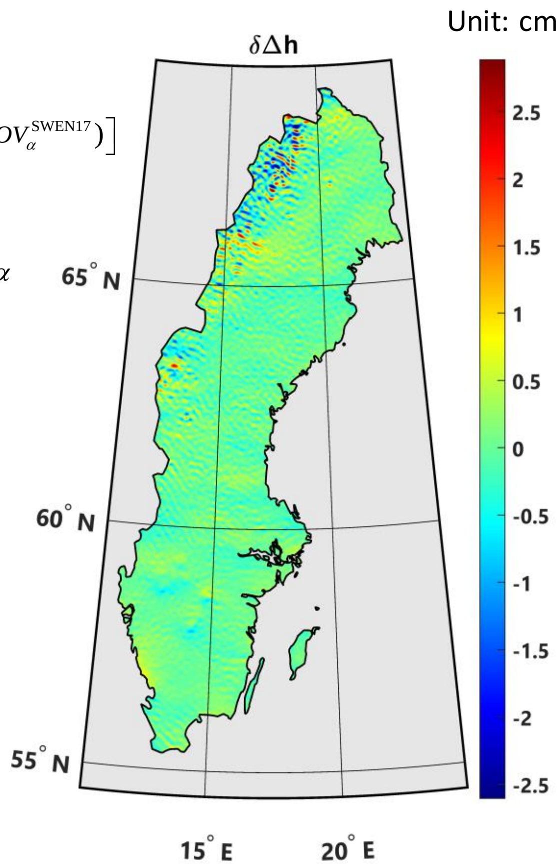
Assuming Azimuth α= 0 ˚, FOV= 46.1 and **4km** flight altitude

$$\Delta\delta h = z\left[\sin(DOV_\alpha^{\mathrm{EGM2008}}) - \sin(DOV_\alpha^{\mathrm{SWEN17}})\right]$$

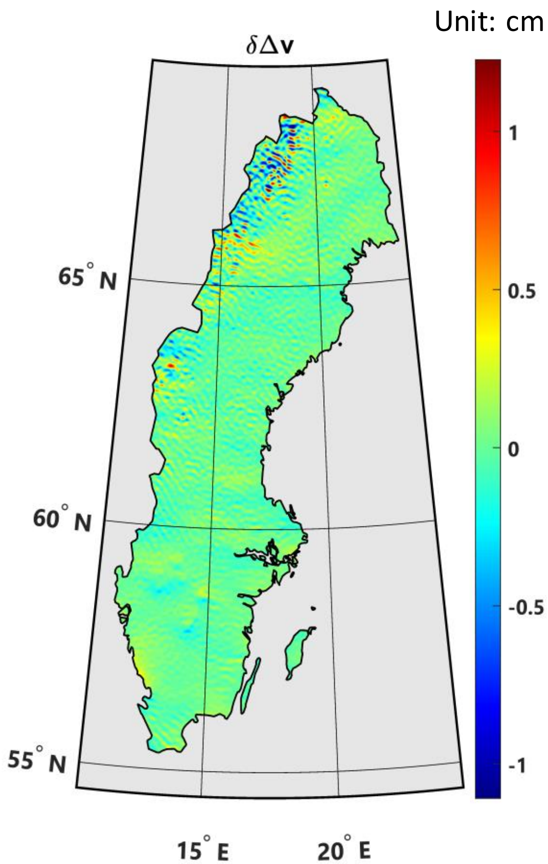
$$\Delta\delta v = z\tan\left(\frac{FOV}{2}\right)\left[\sin(DOV_\alpha^{\mathrm{EGM2008}}) - \sin(DOV_\alpha^{\mathrm{SWEN17}})\right]$$

$$DOV_\alpha^{\mathrm{EGM2008}} = \xi_{\mathrm{EGM2008}}\cos\alpha + \eta_{\mathrm{EGM2008}}\sin\alpha$$

$$DOV_\alpha^{\mathrm{SWEN17}} = \xi_{\mathrm{SWEN17}}\cos\alpha + \eta_{\mathrm{SWEN17}}\sin\alpha$$



< ± 3 cm



< ± 1.5 cm

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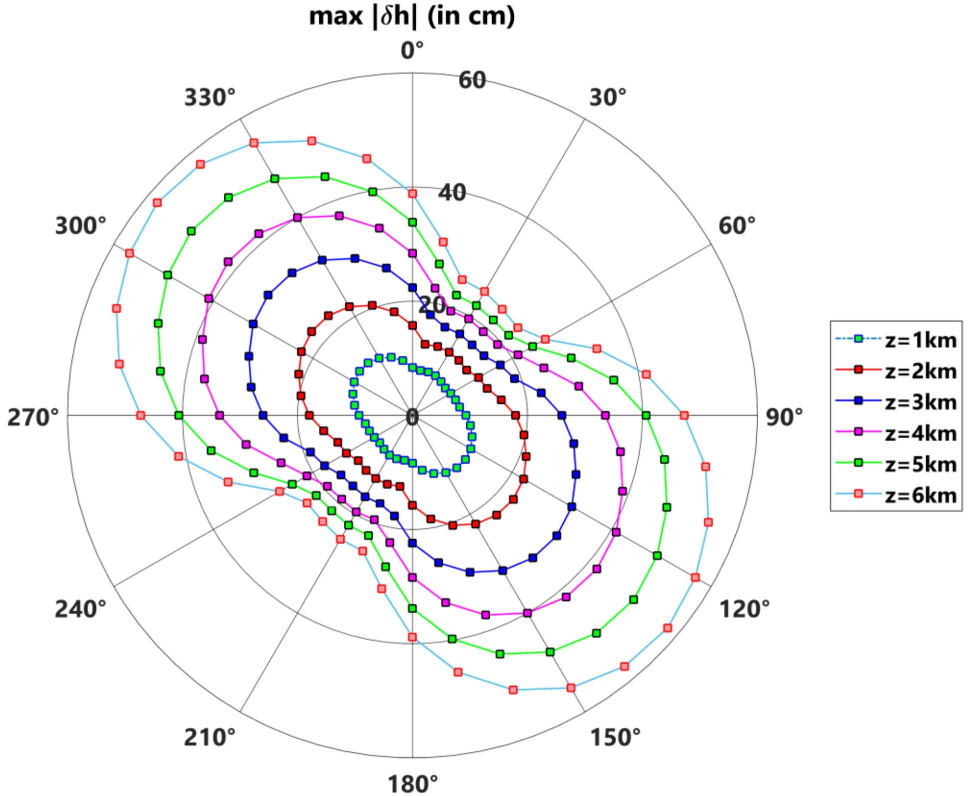
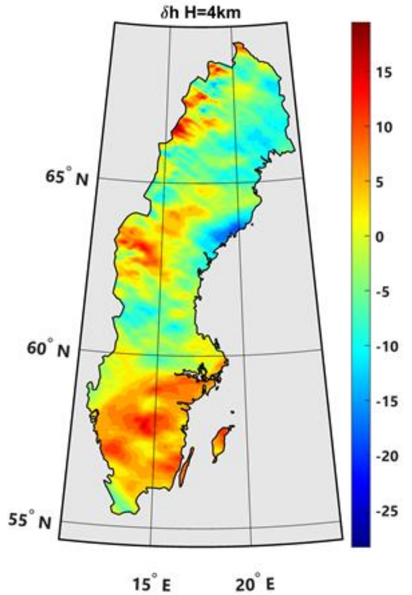
Impact of azimuth-angle variations

$$DOV = \xi \cos\alpha + \eta \sin\alpha$$

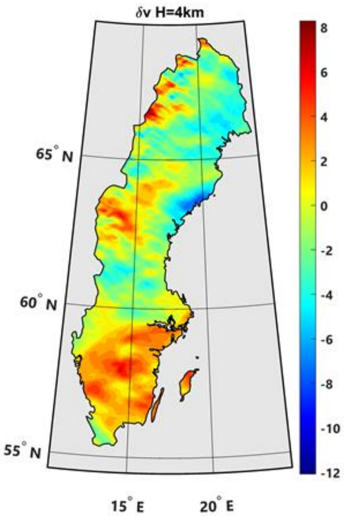
$$\delta h = z \sin(DOV)$$

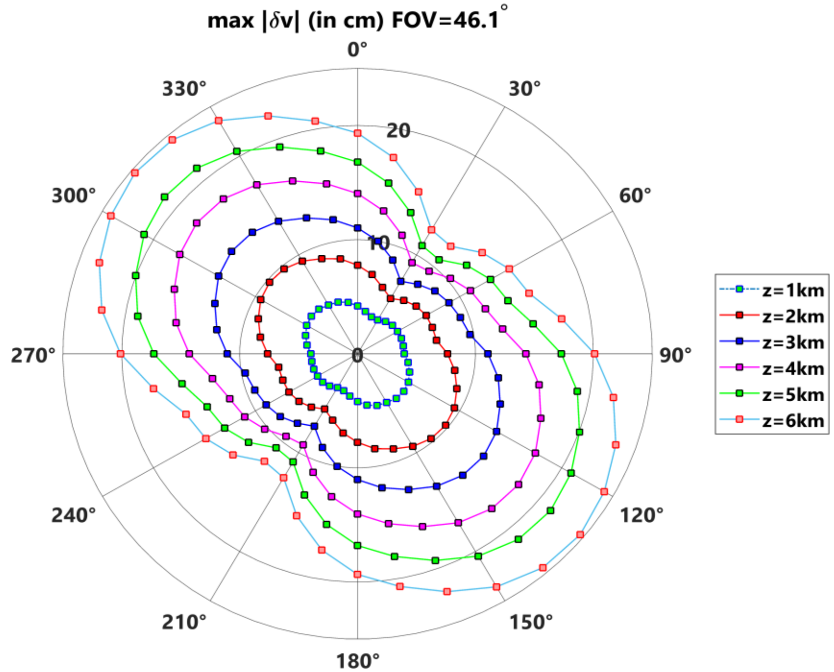


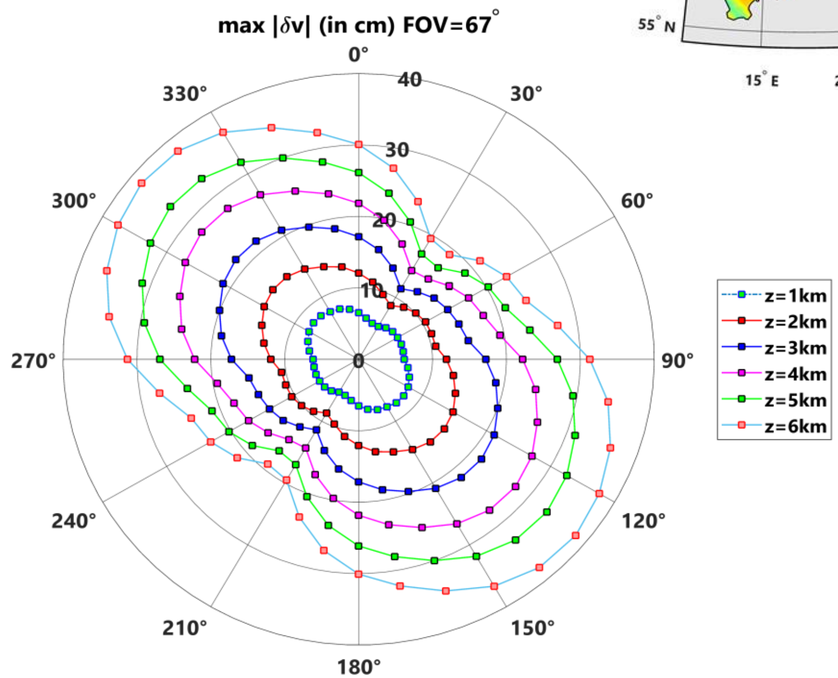


Impact of azimuth-angle variations (FOV= 46.1° and 67°)

$$\delta v = z \tan\left(\frac{FOV}{2}\right)\sin(DOV)$$







Impact of azimuth-angle variations in Götoberg, Jönköping, Dalarna and Norrbotten

Source: https://www.britannica.com/place/Sweden

Location	Latitude	Longitude	Best Azimuth
Göteborg	56°-58°N	12°-13.5°E	~120° (300°)
Jönköping	56°-58°N	13°-16°E	~110° (290°)
Dalarna	61°-63°N	13°-16°E	~150° (330°)
Norrbotten	66°-68°N	17°-20°E	~170° (350°)

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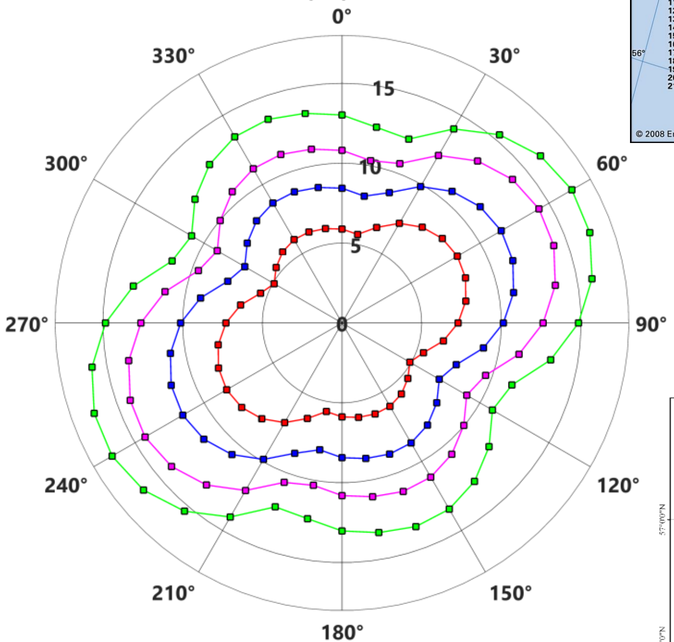
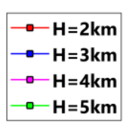
Impact of azimuth-angle variations (FOV= 67˚) in Götoberg

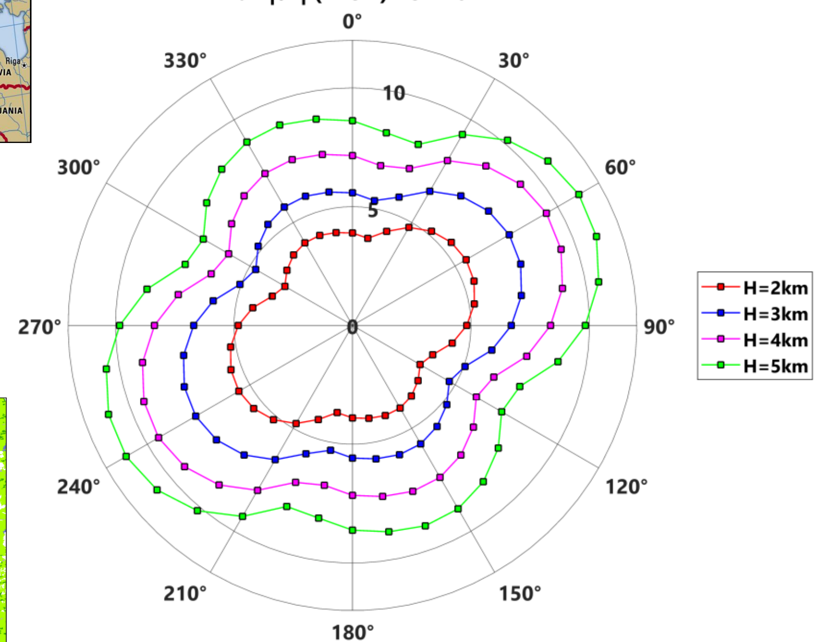
$$\delta h = z\sin(DOV)$$

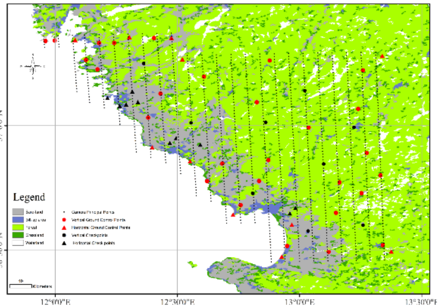
$$\delta v = z\tan\left(\frac{FOV}{2}\right)\sin(DOV)$$

Azimuth = ~120˚(300˚)







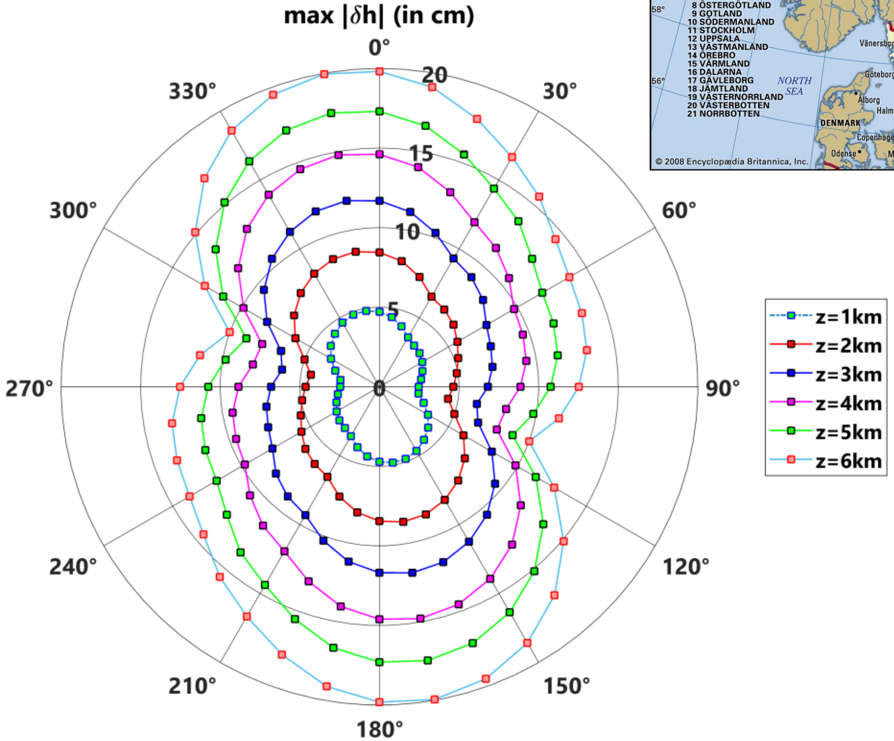


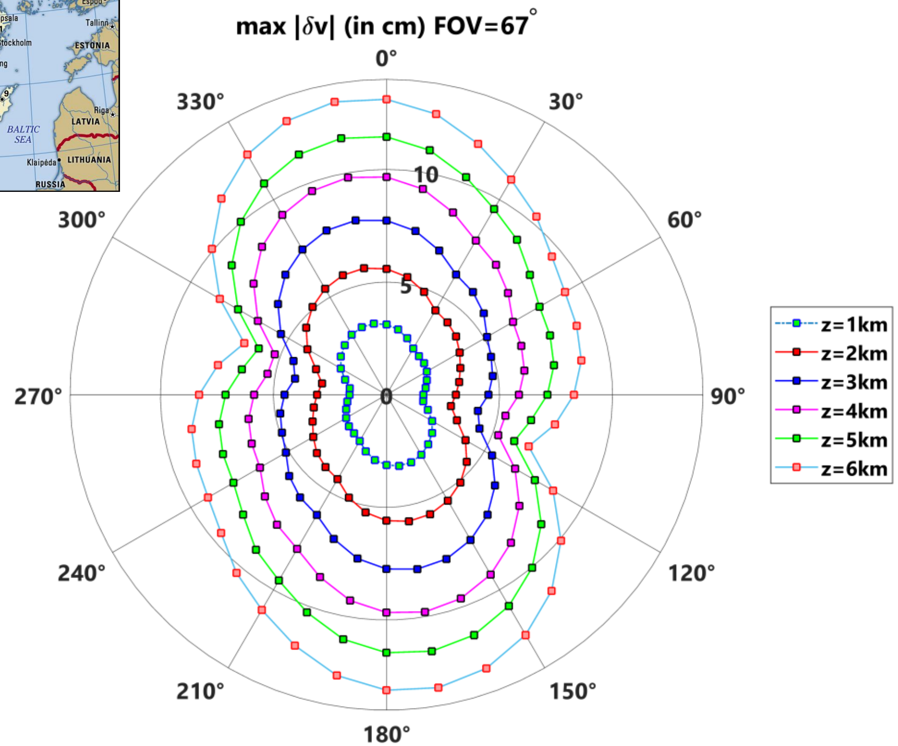
Impact of azimuth-angle variations (FOV= 67°) in Jönköping

$$\delta h = z\sin(DOV)$$

$$\delta v = z\tan\left(\frac{FOV}{2}\right)\sin(DOV)$$

Azimuth = ~110° (290°)





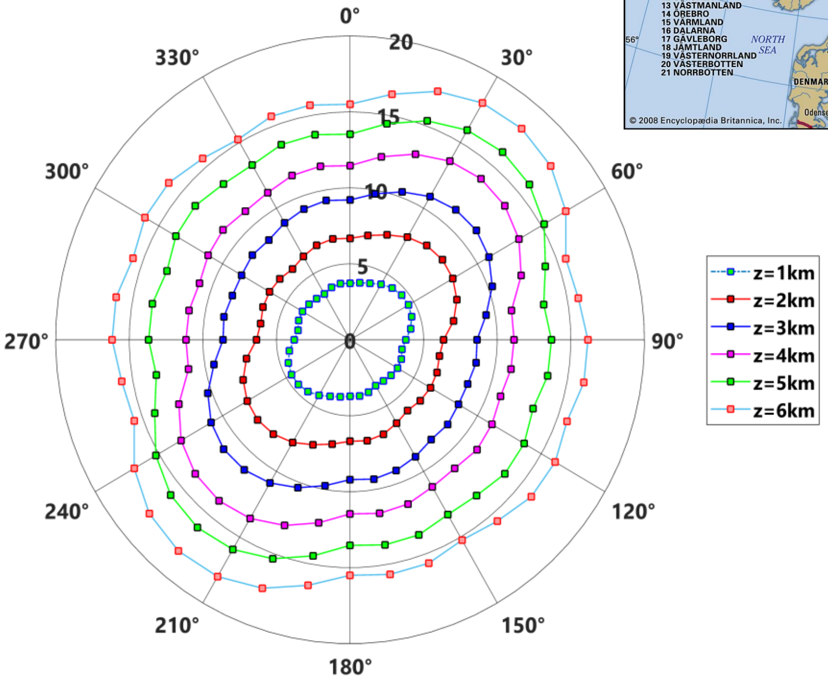
Impact of azimuth-angle variations (FOV= 67°) in Dalarna

$$\delta h = z\sin(DOV)$$

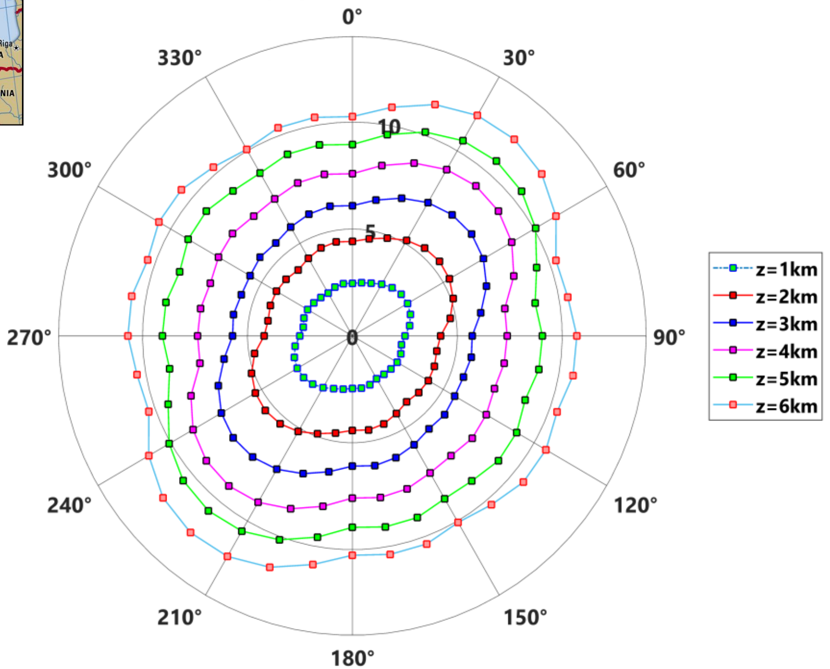
$$\delta v = z\tan\left(\frac{FOV}{2}\right)\sin(DOV)$$

Azimuth = ~150° (330°)

max |δh| (in cm)



max |δv| (in cm) FOV=67°



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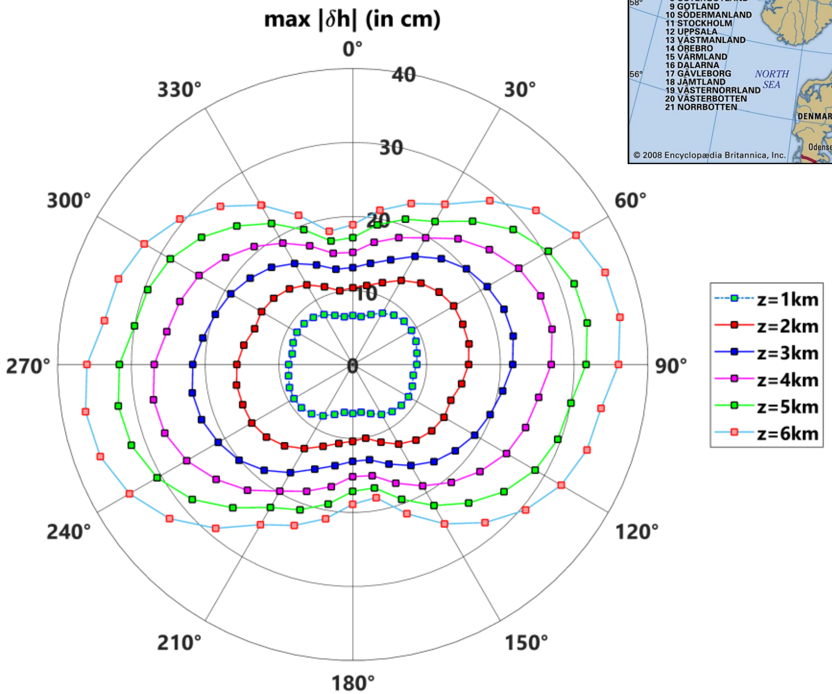
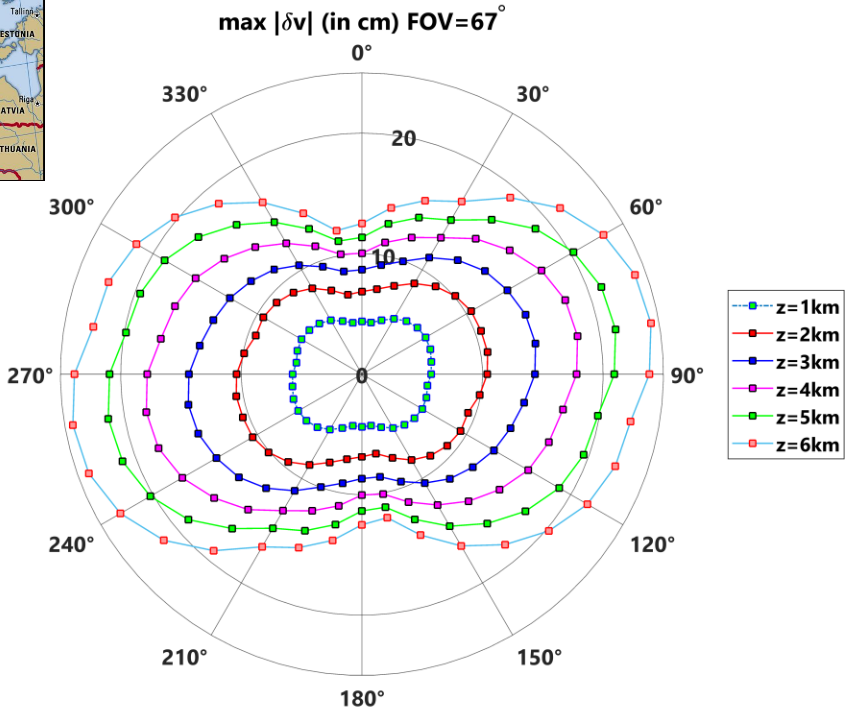
Impact of azimuth-angle variations (FOV= 67˚) in Norrbotten

$$\delta h = z \sin(DOV)$$

$$\delta v = z \tan\left(\frac{FOV}{2}\right)\sin(DOV)$$

Azimuth = ~170˚ (350˚)





Impact of azimuth-angle variations in Götoberg, Jönköping, Dalarna and Norrbotten

Source: https://www.britannica.com/place/Sweden

Location	Latitude	Longitude	Best Azimuth
Göteborg	56°-58°N	12°-13.5°E	~120° (300°)
Jönköping	56°-58° N	13°-16° E	~110° (290°)
Dalarna	61°-63° N	13°-16°E	~150° (330°)
Norrbotten	66°-68° N	17°-20° E	~170° (350°)

Conclusions

- We studied the impact of the geoid slope with respect to the Earth's reference ellipsoid in 3D mapping using aerial photogrammetry in Sweden

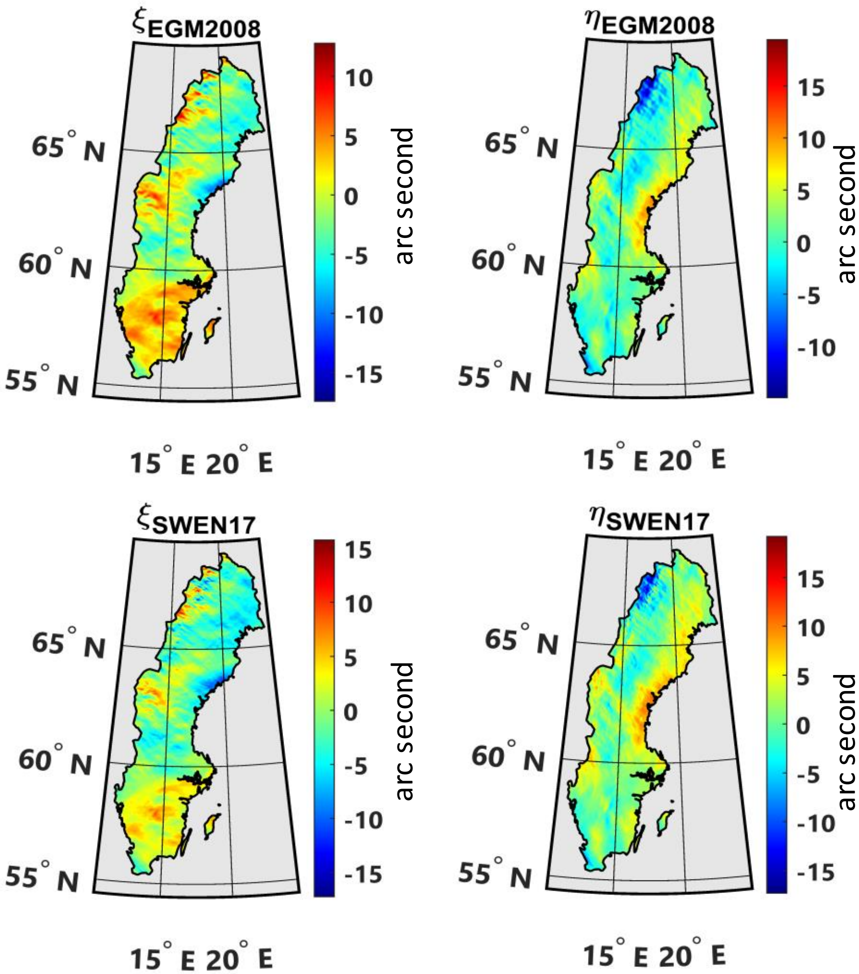
- The influence of the anomalous gravity field (deflection of verticals) in GNSS/INS applications is not ignorable.
  - Latest Applanix company's INS sensor (POS AV 610 model) provides inertial data with high accuracy (about 9" for roll and pitch and 18" for heading (yaw)).

POS AV	510 SPS	510 RTX[3]	510 RTX Post-Processed[4]	510 SmartBase Post-Processed[4]	610 SPS	610 RTX[3]	610 RTX Post-Processed[4]	610 SmartBase Post-Processed[4]
Position (m)	1.5 H 3 V	<0.1 H <0.2 V	<0.1 H <0.2 V	<0.05 H <0.1 V	1.5 H 3 V	<0.1 H <0.2 V	<0.1 H <0.2 V	<0.05 H <0.1 V
Velocity (m/s)	0.050	0.050	0.005	0.005	0.030	0.030	0.0050	0.0050
Roll and Pitch (deg)	0.008	0.008	0.005	0.005	0.005	0.005	0.0025[5]	0.0025[5]
True Heading[2] (deg)	0.070	0.040	0.008	0.008	0.030	0.020	0.0050	0.0050

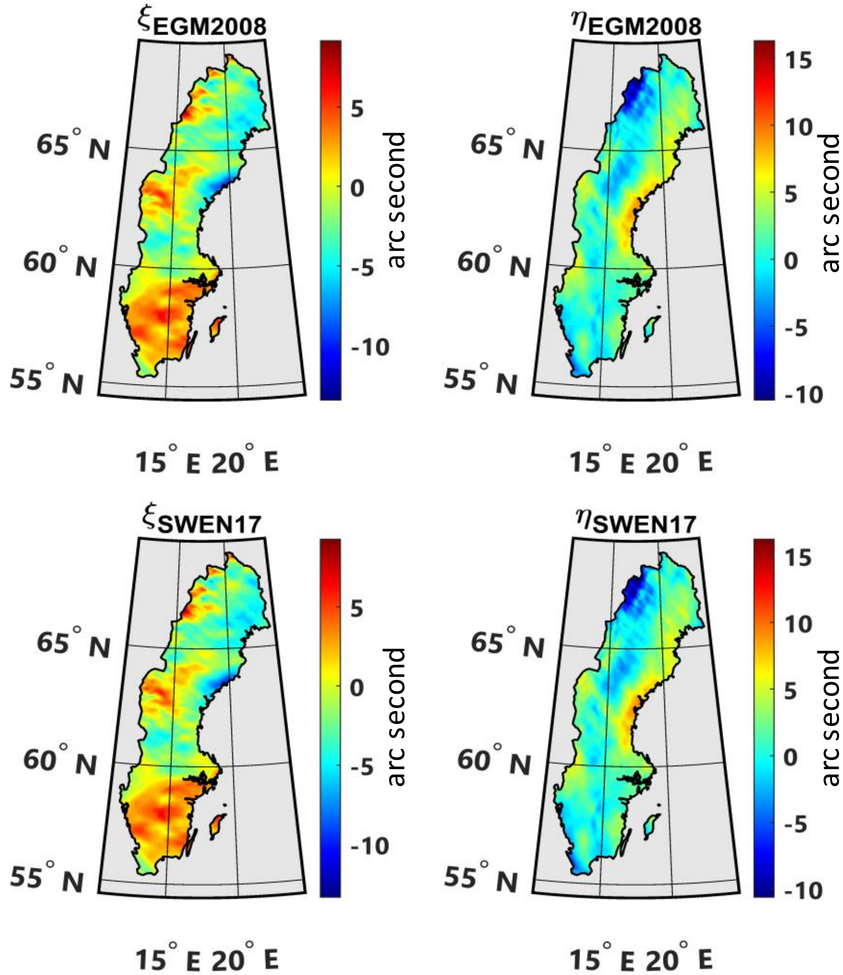
0.0025 (deg) x 3600 = 9"
0.005 (deg) x 3600 = 18"

Results: DOV comparison using SWEN17 and EGM08 at different flight heights

Flight height= 1 km



Flight height= 6 km



Conclusions

- Our results show that the max DOV impact (in Sweden) is about -30 cm and -20 cm in horizontal and vertical components, respectively, considering:
  - z = 4000 m
  - FOV = 67°
  - Azimuth = 0°

- The impact of Azimuth and flight height should be investigated in the planning stage properly.

- The results showed that the calculated DOV using the EGM2008 model is sufficiently precise in Sweden except for the mountainous areas because of the resolution of the EGM2008 model and the topographic signal was not corrected in the EGM2008 model.
  - SWEN17 geoid model is proposed for the rough topography areas

Thank you!

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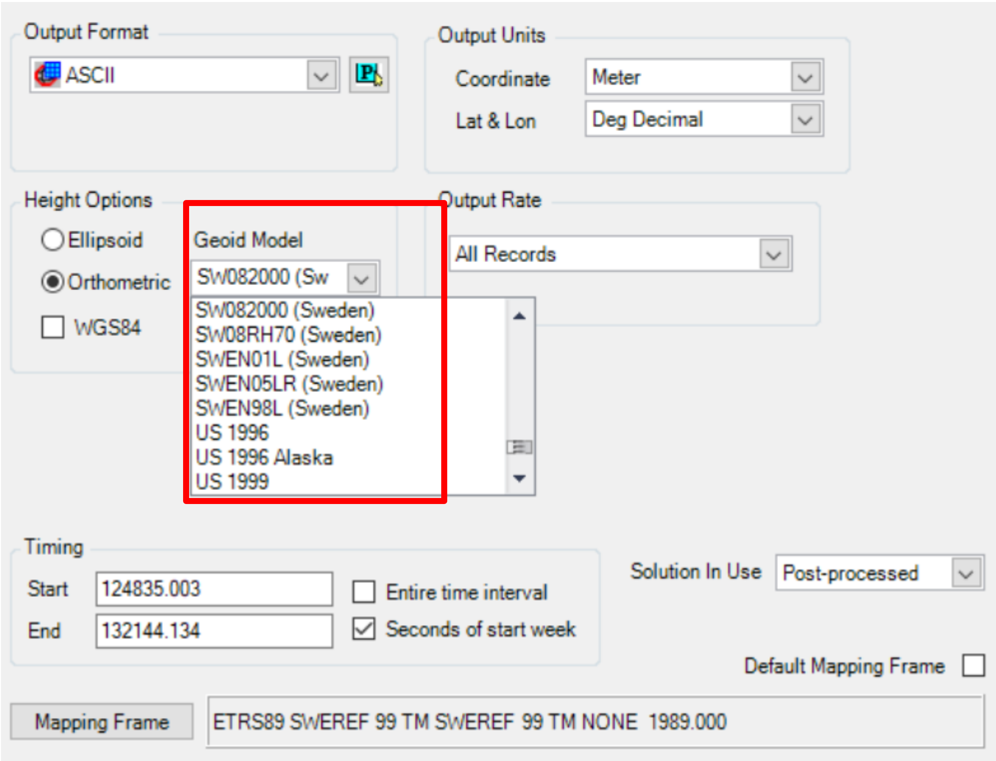
EXTRA SLIDES

Applanix POSPac



$$\xi(r,\theta,\lambda) = -\frac{1}{r}\frac{\partial N}{\partial \theta} = -\frac{1}{r\gamma}\frac{\partial T}{\partial \theta}$$

$$\eta(r,\theta,\lambda) = -\frac{1}{r\sin\theta}\frac{\partial N}{\partial \lambda} = -\frac{1}{r\gamma\sin\theta}\frac{\partial T}{\partial \lambda}$$

EGM
or
regional geoid model?

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Extra slide

$$\mathbf{r}_{Ground} = \mathbf{r}_{GNSS} + \mathbf{R}_{DOV}\mathbf{R}_{INS}\left(\mathbf{r}_{lever\ arm} + s\mathbf{R}_{Boresight}\mathbf{r}_{image}\right)$$

The DOV rotation matrix is the product of two matrices $R_{\mathrm{DOV}} = R_{\mathrm{y}}(\eta)R_{\mathrm{x}}(\xi)$

$$R_{\mathrm{x}}(\xi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\xi) & -\sin(\xi) \\ 0 & \sin(\xi) & \cos(\xi) \end{bmatrix}$$

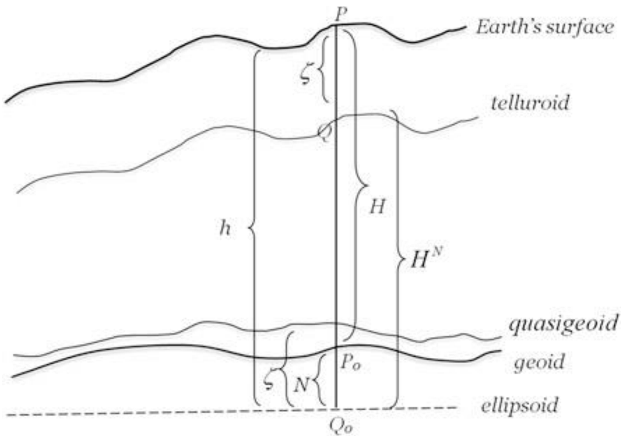
$$R_{\mathrm{y}}(\eta) = \begin{bmatrix} \cos(\eta) & 0 & \sin(\eta) \\ 0 & 1 & 0 \\ -\sin(\eta) & 0 & \cos(\eta) \end{bmatrix}$$

The DOV matrix becomes

$$R_{\mathrm{DOV}} = \begin{bmatrix} \cos(\eta) & \sin(\xi)\sin(\eta) & \sin(\eta)\cos(\xi) \\ 0 & \cos(\xi) & -\sin(\xi) \\ -\sin(\xi) & \cos(\eta)\sin(\xi) & \cos(\xi)\cos(\eta) \end{bmatrix}$$

Geoid vs quasigeoid

The world-wide task of the geodetic community today to determine "the 1-cm geoid" is not easy, in particular in mountainous regions, as there is a major problem stemming from the geoid dependence on the only partly known **topographic mass distribution**, and this problem occurs also in determining orthometric heights.



In 1945 M. S. **Molodensky** (Molodensky et al. 1962) suggested **substituting the geoid** and **orthometric height** with the concepts of the **quasigeoid** and **normal height**, a brilliant idea to avoid the above problem with the topographic mass distribution.

Deflection of vertical components using EGM2008

$$\xi(r,\theta,\lambda) = -\frac{1}{r}\frac{\partial N}{\partial \theta} = -\frac{1}{r\gamma}\frac{\partial T}{\partial \theta}$$

$$\eta(r,\theta,\lambda) = -\frac{1}{r\sin\theta}\frac{\partial N}{\partial \lambda} = -\frac{1}{r\gamma\sin\theta}\frac{\partial T}{\partial \lambda}$$

$$T(r,\theta,\lambda) = \frac{GM}{a}\sum_{n=2}^{n_{\max}}\sum_{m=0}^{n}\left(\frac{a}{r}\right)^{n+1}\left(\bar{c}_{nm}\cos m\lambda + \bar{s}_{nm}\sin m\lambda\right)\bar{P}_{nm}(\cos\theta)$$

$$\xi(r,\theta,\lambda) = \frac{GM}{ar\gamma}\sum_{n=2}^{n_{\max}}\sum_{m=0}^{n}\left(\frac{a}{r}\right)^{n+1}\left(\bar{c}_{nm}\cos m\lambda + \bar{s}_{nm}\sin m\lambda\right)\left(\bar{P}_{nm+1}(\cos\theta) - m\tan\varphi\bar{P}_{nm}(\cos\theta)\right)$$

$$\eta(r,\theta,\lambda) = \frac{GM}{a\gamma r\sin\theta}\sum_{n=2}^{n_{\max}}\sum_{m=0}^{n}\left(\frac{a}{r}\right)^{n+1} m\left(\bar{c}_{nm}\sin m\lambda - \bar{s}_{nm}\cos m\lambda\right)\bar{P}_{nm}(\cos\theta)$$

n	m	c_{nm}	s_{nm}
0	0	1.00000000000E+00	0.00000000000E+00
1	0	0.00000000000E+00	0.00000000000E+00
1	1	0.00000000000E+00	0.00000000000E+00
2	0	-0.484165143790815E-03	0.000000000000000E+00
2	1	-0.206615509074176E-09	0.138441389137979E-08
2	2	0.243938357328313E-05	-0.140027370385934E-05
3	0	0.957161207093473E-06	0.000000000000000E+00
3	1	0.203046201047864E-05	0.248200415856872E-06
3	2	0.904787894809528E-06	-0.619005475177618E-06
3	3	0.721321757121568E-06	0.141434926192941E-05
4	0	0.539965866638991E-06	0.000000000000000E+00
4	1	-0.536157389388867E-06	-0.473567346518086E-06
4	2	0.350501623962649E-06	0.662480026275829E-06
4	3	0.990856766672321E-06	-0.200956723567452E-06
4	4	-0.188519633023033E-06	0.308803882149194E-06
.	.		
.	.		

Spherical harmonics coefficients

$$x = (\mathbb{N} + h)\cos\varphi\cos\lambda$$

$$y = (\mathbb{N} + h)\cos\varphi\sin\lambda$$

$$z = \left[\mathbb{N}(1 - \mathrm{e}^2) + h\right]\sin\varphi$$

$$r = \sqrt{x^2 + y^2 + z^2} \qquad \mathbb{N} = a/\sqrt{1 - e^2\sin^2\varphi}$$

Computation of normal gravity value of a point on Geodetic Reference System 1980 (GRS80) ellipsoid using Somigliana's formula

$$\gamma = \gamma_e \frac{1 + k\sin^2\varphi}{\left(1 - e^2\sin^2\varphi\right)^{1/2}} \qquad k = \frac{b\gamma_p - a\gamma_e}{a\gamma_e}$$

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Upward continuation

- **Using earth's global gravitational model EGM2008 (spectral space)**

$$\xi(r,\theta,\lambda)=\frac{GM}{ar\gamma}\sum_{n=2}^{n_{\max}}\sum_{m=0}^{n}\left(\frac{a}{r}\right)^{n+1}\left(\bar{c}_{nm}\cos m\lambda+\bar{s}_{nm}\sin m\lambda\right)\left(\bar{P}_{nm+1}(\cos\theta)-m\tan\varphi\bar{P}_{nm}(\cos\theta)\right)$$

$$\eta(r,\theta,\lambda)=\frac{GM}{a\gamma r\sin\theta}\sum_{n=2}^{n_{\max}}\sum_{m=0}^{n}\left(\frac{a}{r}\right)^{n+1}m\left(\bar{c}_{nm}\sin m\lambda-\bar{s}_{nm}\cos m\lambda\right)\bar{P}_{nm}(\cos\theta)$$

- **Regional geoid model (using Poisson's Integral)**

$$f_{(r,\theta,\lambda)}=\frac{R\left(r^2-R^2\right)}{4\pi}\int_{\lambda'=0}^{2\pi}\int_{\theta'=0}^{\pi}\frac{f_{(R,\theta',\lambda')}}{l^3}\sin\theta' d\theta' d\lambda' \longrightarrow F\left(\left[\xi \quad \eta\right]_{(k,z)}\right)=F\left(\left[\xi \quad \eta\right]_{(k,0)}\right)\exp^{(-2\pi kz)}$$

$$l=\sqrt{r^2+R^2-2rR\cos\psi}$$

$$r=R+H$$

Heiskanen and Moritz (1967, p. 37) and Sjöberg and Bagherbandi (2017, p. 94)

By using the value of the gradients of [ ] function on the surface, [ ] can be expanded as a **Taylor series** as follows:

Flight altitude — Earth's surface

$$\left[\xi \quad \eta\right]_{(R+H+z,\theta,\lambda)}=\left[\xi \quad \eta\right]_{(R+H,\theta,\lambda)}+\frac{\partial\left[\xi \quad \eta\right]}{\partial r}z+\frac{\partial^2\left[\xi \quad \eta\right]}{\partial r^2}z^2+\ldots$$

by neglecting second and higher order terms, this equation can be written in linear approximation. To determine $\frac{\partial\xi}{\partial r}$ see Heiskanen and Moritz (1967, p.38).

Effect of curvature of the plumb line on ξ

$$\delta\xi = -\int_{H_A}^{H_B} k_x dH \approx 0.17'' \sin 2\varphi \Delta H$$

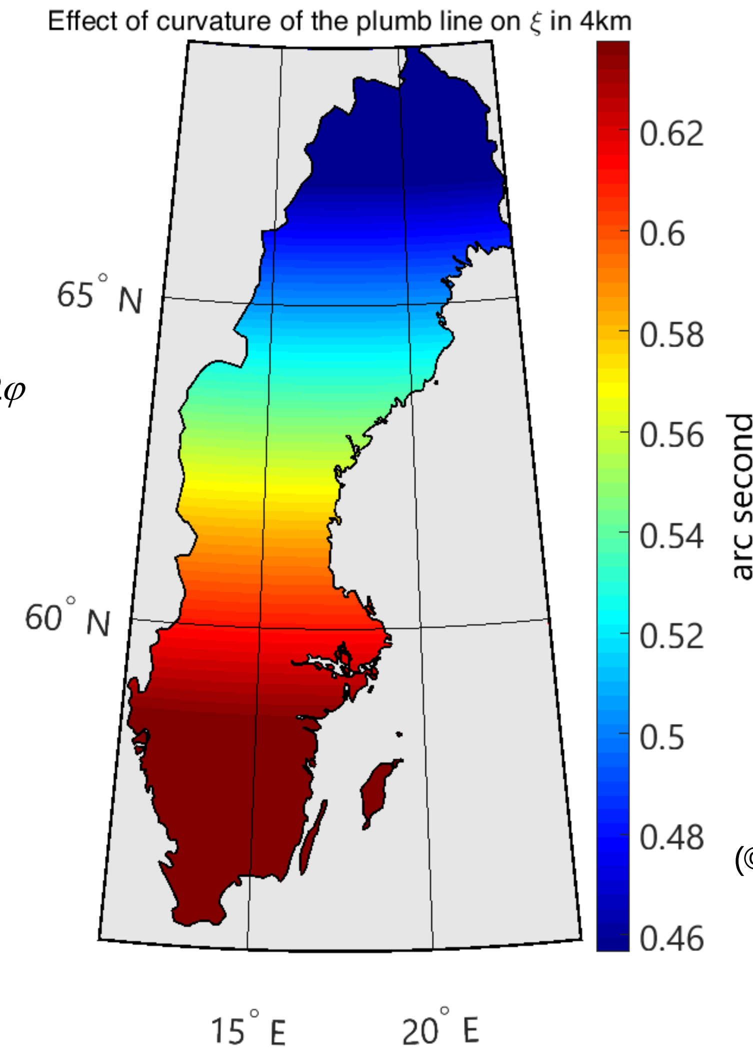
$$k_x = \frac{1}{g}\frac{\partial g}{\partial x}\bigg|_P \approx \frac{1}{\gamma}\frac{\partial \gamma}{\partial x} = \frac{1}{r}\frac{\partial \gamma}{\partial \varphi} = \frac{\gamma_e}{r} f \sin 2\varphi$$

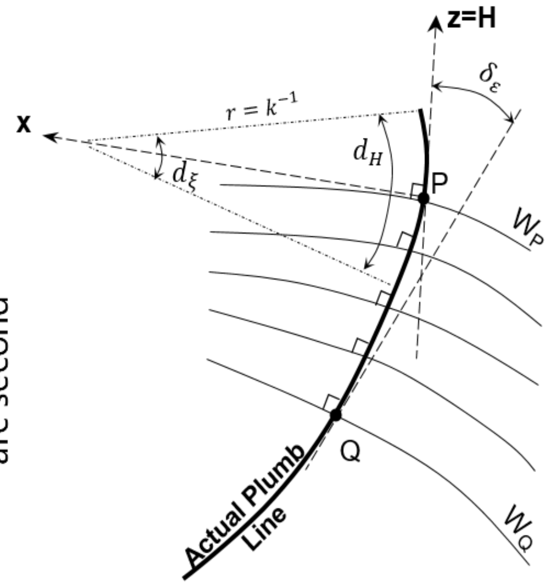
Normal gravity

Actual gravity

$$\gamma = \gamma_e \frac{1 + k \sin^2\varphi}{(1 - e^2 \sin^2\varphi)^{1/2}}$$

Somigliana's formula





Curvature of the plumb line
(© M. Bagherbandi modified after
Vanicek and Krakiwsky 1980)

Comparison of the SWEN17 and EGM2008 models

$$\Delta\delta h = z\left[\sin(DOV_\alpha^{\text{EGM2008}}) - \sin(DOV_\alpha^{\text{SWEN17}})\right]$$

$$\Delta\delta v = z\tan\left(\frac{FOV}{2}\right)\left[\sin(DOV_\alpha^{\text{EGM2008}}) - \sin(DOV_\alpha^{\text{SWEN17}})\right]$$

$$DOV_\alpha^{\text{EGM2008}} = \xi_{\text{EGM2008}}\cos\alpha + \eta_{\text{EGM2008}}\sin\alpha$$

$$DOV_\alpha^{\text{SWEN17}} = \xi_{\text{SWEN17}}\cos\alpha + \eta_{\text{SWEN17}}\sin\alpha$$

			Flight altitudes (km)					
			z = 1	**z = 2**	**z = 3**	**z = 4**	**z = 5**	**z = 6**
$\Delta\delta h$ (α=0°) cm		**Max**	2.58	3.31	3.25	2.88	2.60	2.78
		Mean	0.00	0.00	-0.01	-0.01	-0.01	-0.01
		Min	-2.76	-3.25	-3.05	-2.61	-2.33	-2.14
		STD	0.29	0.37	0.38	0.38	0.37	0.37
$\Delta\delta v$ (α=0°) cm	FOV= 46.1°	**Max**	1.10	1.41	1.38	1.23	1.11	1.18
		Mean	0.00	0.00	0.00	0.00	0.00	-0.01
		Min	-1.18	-1.38	-1.30	-1.11	-0.99	-0.91
		STD	0.13	0.16	0.16	0.16	0.16	0.16
	FOV=67°	**Max**	1.71	2.19	2.15	1.91	1.72	1.84
		Mean	0.00	0.00	0.00	-0.01	-0.01	-0.01
		Min	-1.83	-2.15	-2.02	-1.73	-1.54	-1.42
		STD	0.19	0.25	0.25	0.25	0.25	0.25

Comparison of the DOV component obtained from SWEN17 model and subtraction of astronomical and geodetic coordinates in Stockholm observatory $\varphi = 59^\circ 20' 29.16''$, $\lambda = 18^\circ 03' 16.76''$

$$\xi^{astro-geo} = \Phi - \varphi$$

$$\eta^{astro-geo} = (\Lambda - \lambda)\cos\varphi$$

DOV component	Magnitude	
$\xi^{astro-geo}$	Using Wargentin (1759) estimation for Φ $(59^\circ 20' 31.13'')$	1.97
	Using Cronstrand (1811) estimation for Φ $(59^\circ 20' 34.8'')$	5.64
	Using Selander (1835) estimation for Φ $(59^\circ 20' 33.8'')$	4.64
	Mean value of $\xi^{astro-geo}$	4.08
$\eta^{astro-geo}$		6.65
ξ_{SWEN17}		3.87
η_{SWEN17}		6.57

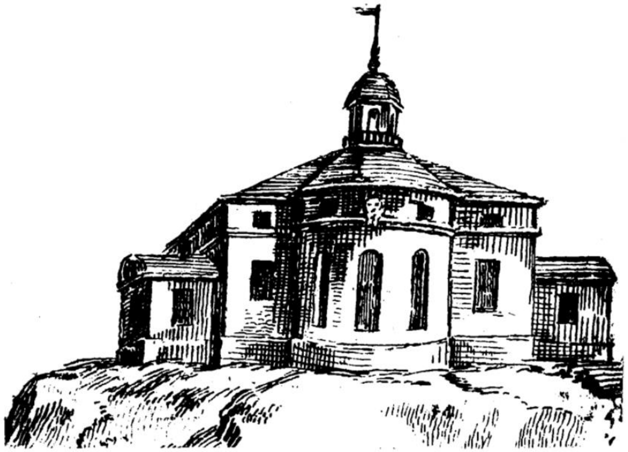


Figure 1. The old observatory of Stockholm (Wargentin, 1761)